Remarks on the Bogoliubov-Valatin transformation

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Abstract. A mistake in the new formulation of the Bogoliubov-Valatin transformation [W.S. Liu, X.P. Li, Eur. Phys. J. D **2**, 1 (1998)] is pointed out and an exact formulation is reconstructed by using the disentangling technique for matrices.

PACS. 03.65.Ca Quantum mechanics: formalism

Recently we have shown that the usual BCS state is a kind of squeezed fermion-pair states by making use of the new formulation of the Bogoliubov-Valatin transformation (BVT) [1]. However, it is much to be regretted that we found that there exists an illogical deduction in this formulation although its occurrence had no effect on the later conclusion. The purposes of the present paper are to highlight this mistake and reconstruct the exact formulation of the BVT.

In what follows we present a brief summary of reference [1]. Consider the general BVT which mixes the annihilation operators $a_{\mathbf{p}\uparrow}(a_{-\mathbf{p}\downarrow})$ and the creation operators $a_{\mathbf{p}\uparrow}^{\dagger}(a_{-\mathbf{p}\downarrow}^{\dagger})$ of a pair of fermions with opposite momenta $(\mathbf{p}, -\mathbf{p})$ and antiparallel spins $(\uparrow,\downarrow)$. It may be written in a form of unitary transformations for the individual fermion operators

$$U_p a_{\mathbf{p}\uparrow} U_p^{\dagger} = \mu_p a_{\mathbf{p}\uparrow} - \nu_p a_{-\mathbf{p}\downarrow}^{\dagger},$$
$$U_p a_{\mathbf{p}\uparrow}^{\dagger} U_p^{\dagger} = \mu_p^* a_{\mathbf{p}\uparrow}^{\dagger} - \nu_p^* a_{-\mathbf{p}\downarrow},$$
$$U_p a_{-\mathbf{p}\downarrow} U_p^{\dagger} = \mu_p a_{-\mathbf{p}\downarrow} + \nu_p a_{\mathbf{p}\uparrow}^{\dagger},$$
$$U_p a_{-\mathbf{p}\downarrow}^{\dagger} U_p^{\dagger} = \mu_p^* a_{-\mathbf{p}\downarrow}^{\dagger} + \nu_p^* a_{\mathbf{p}\uparrow}, \qquad (1)$$

where U_p is a unitary operator, μ_p and ν_p are complex transformation coefficients satisfying the relation:

$$|\mu_p|^2 + |\nu_p|^2 = 1. \qquad (2)$$

If we assume that all U_p, μ_p, and ν_p are functions of a certain real parameter x, after tedious calculation we obtain

$$\frac{\partial U_p}{\partial x} U_p^{\dagger} = \zeta_p^*(x) K_{p+} - \zeta_p(x) K_{p-} - 2\kappa_p(x) K_{p0}, \qquad (3)$$

where

$$\zeta_p(x) = \mu_p \frac{\partial \nu_p^*}{\partial x} - \nu_p^* \frac{\partial \mu_p}{\partial x},$$
$$\kappa_p(x) = \mu_p^* \frac{\partial \mu_p}{\partial x} + \nu_p \frac{\partial \nu_p^*}{\partial x} = -\kappa_p^*(x), \qquad (4)$$

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and

$$K_{p+} = a_{\mathbf{p}\uparrow}^{\dagger} a_{-\mathbf{p}\downarrow}^{\dagger}, \qquad K_{p-} = a_{-\mathbf{p}\downarrow} a_{\mathbf{p}\uparrow},$$
$$K_{p0} = \frac{1}{2}(a_{\mathbf{p}\uparrow}^{\dagger} a_{\mathbf{p}\uparrow} + a_{-\mathbf{p}\downarrow}^{\dagger} a_{-\mathbf{p}\downarrow} - 1). \qquad (5)$$

is a two-mode realization of the $SU(2)$ Lie algebra, which satisfies the commutation relation

$$[K_{p-}, K_{p+}] = -2K_{p0}, \qquad [K_{p0}, K_{p\pm}] = \pm K_{p\pm}. \qquad (6)$$

Since the right-hand side of equation (3) includes a complicated algebra of operators, which do not commute with one another, it is incorrect that the solution of the unitary operator $U_p(x)$ be obtained from a straightforward integral (as was done for Eq. (7) in Ref. [1]). To get the correct solution we will take the disentangling technique for matrices proposed by Fisher, Nieto and Sandberg [2], and Gilmore [3] here. Using the realization of the Pauli matrices of the $SU(2)$ Lie algebra

$$K_{p+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad K_{p-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K_{p0} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, (7)$$

equation (3) then can be represented by

$$\frac{\partial U_p}{\partial x} U_p^{\dagger} = \begin{pmatrix} \kappa_p^* & \zeta_p^* \\ -\zeta_p & \kappa_p \end{pmatrix} = \begin{pmatrix} \mu_p \dot{\mu}_p^* + \nu_p^* \dot{\nu}_p & \mu_p^* \dot{\nu}_p - \nu_p \dot{\mu}_p^* \\ -\mu_p \dot{\nu}_p^* + \nu_p^* \dot{\mu}_p & \mu_p^* \dot{\mu}_p + \nu_p \dot{\nu}_p^* \end{pmatrix} = \begin{pmatrix} \dot{\mu}_p^* & \dot{\nu}_p \\ -\dot{\nu}_p^* & \dot{\mu}_p \end{pmatrix} \begin{pmatrix} \mu_p & -\nu_p \\ \nu_p^* & \mu_p^* \end{pmatrix}, \qquad (8)$$

where the dot denotes the differentiation with respect to x. From equation (8) it is evident that

$$U_p(x) = \begin{pmatrix} \mu_p^* & \nu_p \\ -\nu_p^* & \mu_p \end{pmatrix} = \begin{pmatrix} 1 & \frac{\nu_p}{\mu_p} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\mu_p} & 0 \\ 0 & \mu_p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\nu_p^*}{\mu_p} & 1 \end{pmatrix} = \exp\left[\frac{\nu_p}{\mu_p} K_{p+}\right] \exp[-2\ln\mu_p K_{p0}] \exp\left[-\frac{\nu_p^*}{\mu_p} K_{p-}\right], \qquad (9)$$

which is the normal-order form of $U_p(x)$. Another expression of $U_p(x)$ represented as a single exponential including $K_{pj}(j = +, -, 0)$ can be formally written as

$$U_p(x) = \exp[c_{p+}(x)K_{p+} + c_{p-}(x)K_{p-} + c_{p0}(x)K_{p0}], \tag{10}$$

where $c_{pj}(x)$ are scalar functions of x to be determined. Inserting equation (7) into expression (10) we obtain

$$U_p(x) = \exp\begin{pmatrix} \frac{c_{p0}}{2} & c_{p+} \\ c_{p-} & -\frac{c_{p0}}{2} \end{pmatrix} = \sum_n \frac{1}{(2n)!} \begin{pmatrix} \frac{c_{p0}}{2} & c_{p+} \\ c_{p-} & -\frac{c_{p0}}{2} \end{pmatrix}^n$$

$$+ \sum_n \frac{1}{(n+1)!} \begin{pmatrix} \frac{c_{p0}}{2} & c_{p+} \\ c_{p-} & -\frac{c_{p0}}{2} \end{pmatrix}^{(n+1)} =$$

$$\begin{pmatrix} \cosh\Theta_p + \frac{c_{p0}}{2\Theta_p}\sinh\Theta_p & \frac{c_{p+}}{\Theta_p}\sinh\Theta_p \\ \frac{c_{p-}}{\Theta_p}\sinh\Theta_p & \cosh\Theta_p - \frac{c_{p0}}{2\Theta_p}\sinh\Theta_p \end{pmatrix}, \tag{11}$$

where $\Theta_p = \left[\left(\frac{c_{p0}}{2}\right)^2 + c_{p+}c_{p-}\right]^{1/2}$. Comparison of equations (11) with (9) yields

$$\mu_p = \cosh\Theta_p - \frac{c_{p0}}{2\Theta_p}\sinh\Theta_p,$$

$$\nu_p = \frac{c_{p+}}{\Theta_p}\sinh\Theta_p, \tag{12}$$

and $c_{p-} = -c_{p+}^*, c_{p0} = -c_{p0}^*$. In order to solve c_{pj} as functions of μ_p and ν_j, or equivalently κ_p and ζ_p, substituting equation (12) into equation (4), after some manipulations we obtain a set of differential equations

$$\frac{c_{p+}^*\dot{\Theta}_p}{\Theta_p} + \left(\frac{c_{p+}^*\dot{c}_{p0} - c_{p0}\dot{c}_{p+}^*}{2\Theta_p}\right)$$

$$\times \sinh^2\Theta_p + \overline{\left(\frac{\dot{c}_{p+}^*}{2\Theta_p}\right)}\sinh 2\Theta_p = \zeta_p,$$

$$-\frac{c_{p0}\dot{\Theta}_p}{2\Theta_p} - \overline{\left(\frac{\dot{c}_{p0}}{4\Theta_p}\right)}\sinh 2\Theta_p = \kappa_p. \tag{13}$$

It is easy to see that one particular solution to equation (13) is

$$c_+(x) = |c_+(x)|e^{i\chi},$$

$$c_-(x) = -|c_+(x)|e^{-i\chi},$$

$$c_0(x) = i|c_0(x)|, \tag{14}$$

where χ does not independ on x and $|c_{p+}(x)| = k|c_{p0}(x)|$, k is a real constant. Then

$$\mu_p(x) = \cos\Xi_p(x) - \frac{1}{(k^2 + \frac{1}{4})^{1/2}}\sin\Xi_p(x),$$

$$\nu_p(x) = \frac{ke^{i\chi}}{(k^2 + \frac{1}{4})^{1/2}}\sin\Xi_p(x), \tag{15}$$

$$U_p(x) = \exp\left\{\int_{x_0}^x \zeta_p^*(x')\mathrm{d}x' K_{p+}\right.$$

$$\left. -\int_{x_0}^x \zeta_p(x')\mathrm{d}x' K_{p-} - 2\int_{x_0}^x \kappa_p(x')\mathrm{d}x' K_{p0}\right\}, \tag{16}$$

where $\Xi_p = (k^2 + \frac{1}{4})^{1/2}\left|-2\int_{x_0}^x \kappa_p(x')\mathrm{d}x'\right|$.

Another particular solution can be found to be

$$c_{p+}(x) = |c_{p+}(x)|e^{i\chi}, \qquad c_{p0}(x) = 0, \tag{17}$$

where χ is also independent of x. Then

$$\mu_p(x) = \cos\left|\int_{x_0}^x \zeta_p^*(x')\mathrm{d}x'\right|,$$

$$\nu_p(x) = e^{i\chi}\sin\left|\int_{x_0}^x \zeta_p^*(x')\mathrm{d}x'\right|, \tag{18}$$

$$U_p(x) = \exp\left\{\int_{x_0}^x \zeta_p^*(x')\mathrm{d}x' K_{p+} - \int_{x_0}^x \zeta_p(x')\mathrm{d}x' K_{p-}\right\}. \tag{19}$$

It should be noted that although equation (3) also admits the solution of the type (16), whose form is the same as equation (8) in reference [1] which is free from any constraint, the functions $c_{pj}(x)$ must obey the restricted condition (14) and additional demands. While equations (22) and (21) in reference [1] are just the proper cases of the equations (18) and (19) with $\chi = 0$, $x = t$, and $x_0 = 0$.

Finally we wish to point out that equation (3) appears to be a time-evolution equation in quantum mechanics [4], the solution of this type of operator differential equations had been widely studied [5–9]. The present paper provides an interesting instance for getting the correct solution with the matrix derivation. However, some carelessness still happened occasionally. For instance, equation (1-133) of reference [10] represents an evolution equation of a one-dimensional linear harmonic oscillator with mass $m = 1$ and an electric charge e that is interacting with an external homogeneous constant electric field $\mathcal{E}$,

$$\frac{1}{i}\frac{\partial U}{\partial t} - e\mathcal{E}\left(q\cos\omega t + \frac{p}{\omega}\sin\omega t\right)U = 0, \tag{20}$$

where q and p are the time-independent canonical

operators, respectively. According to Magnus [5], if certain unspecified conditions of convergence are satisfied, $U(t)$ can be written in the form

$$U(t) = \exp \Omega(t), \qquad (21)$$

where $\Omega(t)$ is an infinite series

$$\Omega(t) = \int_0^t A(\tau)\mathrm{d}\tau + \frac{1}{2}\int_0^t \left[A(\tau), \int_0^\tau A(\sigma)\mathrm{d}\sigma\right] \mathrm{d}\tau$$

$$+\frac{1}{4}\int_0^t \left[A(\tau), \int_0^\tau \left[A(\sigma), \int_0^\sigma A(\rho)\mathrm{d}\rho\right] \mathrm{d}\sigma\right] \mathrm{d}\tau$$

$$+\frac{1}{12}\int_0^t \left[\left[A(\tau), \int_0^\tau A(\sigma)\mathrm{d}\sigma\right] \int_0^\tau A(\sigma)\mathrm{d}\sigma\right] \mathrm{d}\tau + \cdots, (22)$$

with

$$A(t) = ie\mathcal{E}\left(q\cos\omega t + \frac{p}{\omega}\sin\omega t\right). \qquad (23)$$

As it happens that the following commutation relation is identically satisfied for all values of t ,

$$\left[A(t), \int_0^t A(\tau)\mathrm{d}\tau\right] = 0, \qquad (24)$$

$U(t)$ then becomes simply

$$U(t) = \int_0^t A(\tau)\mathrm{d}\tau =$$

$$\exp\left[ie\mathcal{E}\left(\frac{q}{\omega}\sin\omega t - \frac{p}{\omega^2}\cos\omega t + \frac{p}{\omega^2}\right)\right], \;\; (25)$$

which is just the formal solution given in reference [10] (Eq. (1–134) in Ref. [10]). However, using $[q,p] = i$, equation (24) will lead to $\cos\omega t = 1$, which remains true only for the trivial case of $t = 0$. Hence, equation (25) is not a correct expression of the solution to equation (20). On the contrary, it is easily verified that the two particular solutions (16) and (19) to our formalism will identically satisfy the similar relation (24) provided $A(t)$ is replaced by the right-hand side of equation (3).

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